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Non-commuting graph of a group ☆

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Abstract

Let G be a non-abelian group and let $Z(G)$ be the center of G. Associate a graph Γ_G (called non-commuting graph of G) with G as follows: Take $G\backslash Z(G)$ as the vertices of Γ_G and join two distinct vertices x and y, whenever $xy \neq yx$. We want to explore how the graph theoretical properties of Γ_G can effect on the group theoretical properties of G. We conjecture that if G and H are two non-abelian finite groups such that $\Gamma_G \cong \Gamma_H$, then $|G| = |H|$. Among other results we show that if G is a finite non-abelian nilpotent group and H is a group such that $\Gamma_G \cong \Gamma_H$ and $|G| = |H|$, then H is nilpotent.

Keywords: Non-commuting graph; Finite group

1. Introduction

The study of algebraic structures, using the properties of graphs, becomes an exciting research topic in the last twenty years, leading to many fascinating results and questions. There are many papers on assigning a graph to a ring or group and investigation of algebraic properties of ring or group using the associated graph, for instance, see [5–7,9,10,31]. In the present article to

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any non-abelian group G we assign a graph and investigate algebraic properties of the group using the graph theoretical concepts. Before starting let us introduce some necessary notation and definitions.

Let G be a group. One can associate a graph to G in many different ways (see, for example, [11,17,27,28,31,39]). Here we consider the following way: Let $Z(G)$ be the center of G. Associate a graph Γ_G with G as follows: Take $G\backslash Z(G)$ as the vertices of Γ_G and join two distinct vertices x and y whenever $xy \neq yx$. Note that if G is abelian, then Γ_G is the null graph. The non-commuting graph Γ_G was first considered by Paul Erdös, when he posed the following problem in 1975 [27]: Let G be a group whose non-commuting graph has no infinite complete subgraph. Is it true that there is a finite bound on the cardinalities of complete subgraphs of Γ_G? B.H. Neumann [27] answered positively Erdös' question. We later propose a dual of Erdös' problem in some sense. Erdös' problem and Neumann's answer were the origin of many similar questions and many people considered various kind of questions which were in similar nature (see, for example, [1–4,14,15,24–26]). The main object of this paper is to study how the graph theoretical properties of Γ_G effect on the group theoretical properties of G. We call Γ_G the *non-commuting graph* of G. Also we study what group properties of two non-abelian groups with the isomorphic non-commuting graphs are always the same.

We consider simple graphs which are undirected, with no loops or multiple edges. For any graph Γ, we denote the sets of the vertices and the edges of Γ by $V(\Gamma)$ and $E(\Gamma)$, respectively. The degree $d_\Gamma(v)$ of a vertex v in Γ is the number of edges incident to v and if the graph is understood, then we denote $d_\Gamma(v)$ simply by $d(v)$. The order of Γ is defined $|V(\Gamma)|$ and its maximum and its minimum degrees will be denoted, respectively, by $\Delta(\Gamma)$ and $\delta(\Gamma)$. A graph Γ is *regular* if the degrees of all vertices of Γ are the same. A subset X of the vertices of Γ is called a *clique* if the induced subgraph on X is a complete graph. The maximum size of a clique in a graph Γ is called the *clique number* of Γ and denoted by $\omega(\Gamma)$. A subset X of the vertices of Γ is called an *independent set* if the induced subgraph on X has no edges. The maximum size of an independent set in a graph Γ is called the *independence number* of Γ and denoted by $\alpha(\Gamma)$. Let $k > 0$ be an integer. A *k-vertex coloring* of a graph Γ is an assignment of k colors to the vertices of Γ such that no two adjacent vertices have the same color. The *vertex chromatic number* $\chi(\Gamma)$ of a graph Γ, is the minimum k for which Γ has a k-vertex coloring. A *path* P is a sequence $v_0e_1v_1e_2\cdots e_kv_k$ whose terms are alternately distinct vertices and distinct edges, such that for any i, $1 \leqslant i \leqslant k$, the ends of e_i are v_{i-1} and v_i. In this case P is called a path between v_0 and v_k. The number k is called the *length* of P. If v_0 and v_k are adjacent in Γ by an edge e_{k+1}, then $P \cup \{e_{k+1}\}$ is called a *cycle*. The *length* of a cycle defined the number of its edges. The length of the shortest cycle in a graph Γ is called *girth* of Γ and denoted by *girth*(Γ). A *Hamilton cycle* of Γ is a cycle that contains every vertex of Γ. If v and w are vertices in Γ, then $d(v,w)$ denotes the length of the shortest path between v and w. The largest distance between all pairs of the vertices of Γ is called the *diameter* of Γ, and is denoted by *diam*(Γ). A graph Γ is *connected* if there is a path between each pair of the vertices of Γ. The *vertex connectivity*, $\kappa(\Gamma)$, of a connected graph Γ is the smallest number of vertices whose removal disconnect Γ. A subset S of the vertices of a connected graph Γ is called a *cut set*, if $\Gamma\backslash S$ is not a connected graph. For a graph Γ and a subset S of the vertex set $V(\Gamma)$, denote by $N_\Gamma[S]$ the set of vertices in Γ which are in S or adjacent to a vertex in S. If $N_\Gamma[S] = V(\Gamma)$, then S is said to be a dominating set (of vertices in Γ). The *domination number* of a graph Γ, denoted by $\gamma(\Gamma)$, is the minimum size of a dominating set of the vertices in Γ. A *planar* graph is a graph that can be embedded in the plane so that no two edges intersect geometrically except at a vertex which both are incident. We denote the symmetric group and the alternating group on n letters by S_n

and A_n, respectively. Also Q_8 and D_{2n} are used for the quaternion group with 8 elements and the dihedral group of order $2n$ ($n > 2$), respectively. If $n > 0$ is an integer and q is a prime power, then we denote by $\mathrm{PSL}(n,q)$, $\mathrm{PGL}(n,q)$, $\mathrm{SL}(n,q)$ and $\mathrm{GL}(n,q)$, the projective special linear group, the projective general linear group, the special linear group and the general linear group of degree n over the finite field of size q, respectively. Let G be a group, H a subgroup of G and let $x \in G$. Then $C_G(x)$ and $N_G(H)$ are used for the centralizer of x and the normalizer of H in G, respectively. A set $\mathcal{P} = \{H_1, \ldots, H_n\}$ of subgroups H_i $(i = 1, \ldots, n)$ is said to be a *partition* of G if every non-identity element $x \in G$ belongs to one and only one subgroup $H_i \in \mathcal{P}$.

In Section 2 of the paper, we study some graph properties of the non-commuting graph Γ_G of a non-abelian group G. We see that Γ_G is always connected, its diameter and girth are always 2 and 3, respectively. If G is finite, then Γ_G is Hamiltonian and it is planar if and only if G is finite of order 6 or 8. Regularity of Γ_G, when G is finite, is dealt with. Some results about cut sets in Γ_G are established. It is shown when G belongs to certain classes of groups, the finiteness of all independent sets of Γ_G, implies the existence of a finite bound on the size of all its independent sets. Periodic groups whose non-commuting graphs have domination number 1 are completely characterized. Some results concerning the finiteness of the domination number are proved, e.g., it is shown that if H is a subgroup of G of finite index such that $\gamma(\Gamma_H)$ is finite, then $\gamma(\Gamma_G)$ is finite. The domination number of some classical groups are found.

In Section 3, we concentrate on the following conjecture:

Conjecture 1.1. *Let G and H be two non-abelian finite groups such that $\Gamma_G \cong \Gamma_H$. Then $|G| = |H|$.*

We prove that if Conjecture 1.1 is true for solvable AC-groups, then it is true for all groups, where a group G is called an AC-group if $C_G(x)$ is abelian for all $x \in G \backslash Z(G)$. Conjecture 1.1 is proved when one of the groups in the conjecture is S_n, A_n, D_{2n} or a non-solvable AC-group. It is also proved for some other groups.

It is natural to ask what group properties can be inherited via the non-commuting graphs, i.e.:

Question 1.2. *For which group property $\mathcal{P}$, if G and H are two non-abelian groups such that $\Gamma_G \cong \Gamma_H$, and G has the group property $\mathcal{P}$, then H has also $\mathcal{P}$?*

If Conjecture 1.1 is true, then we show that Question 1.2 is true for the property of being nilpotent. Also it will easily show that the latter question is always true for the property of being finite.

In Section 4, we find the chromatic and clique number of some groups.

In Section 5, we give some groups with unique non-commuting graph, i.e. groups G with the property that if $\Gamma_G \cong \Gamma_H$ for some group H, then $G \cong H$. As expected, and as we shall show, the non-commuting graph of a group, in general, is not unique and there are non-isomorphic groups with the same non-commuting graphs. It is shown that the Suzuki simple groups and $\mathrm{PSL}(2, 2^n)$ ($n > 2$) have unique non-commuting graphs. In view of these results, we state the following conjecture:

Conjecture 1.3. *Let S be a finite non-abelian simple group and G is a group such that $\Gamma_G \cong \Gamma_S$. Then $G \cong S$.*

2. Some properties of non-commuting graphs

Proposition 2.1. *For any non-abelian group G, $diam(\Gamma_G) = 2$. In particular, Γ_G is connected. Also the girth of Γ_G equals* 3.

Proof. Let x and y be two distinct vertices of Γ_G. If x and y are adjacent, then $d(x, y) = 1$. Thus we may assume that $xy = yx$. Since x, y are non-central, there exist $x', y' \in V(\Gamma_G)$ such that $\{x, x'\}$ and $\{y, y'\}$ are edges. If y and x' or x and y' are adjacent then $d(x, y) = 2$. Otherwise the vertex $x'y'$ is adjacent to both x and y and again $d(x, y) = 2$. Let $diam(\Gamma_G) = 1$, and a be a non-central element of G. So we have $a = a^{-1}$ and if b is a central element, then ab is non-central, thus $(ab)^2 = 1$, which implies that $b^2 = 1$. Thus G is abelian, a contradiction.

For every edge $\{x, y\}$ of Γ_G, $\{x, y, xy\}$ is a triangle. Hence the girth of Γ_G is 3. □

Proposition 2.2. *The non-commuting graph of every non-abelian finite group is Hamiltonian.*

Proof. First note that the degree of any vertex x in the non-commuting graph Γ_G of a non-abelian group G is equal to $|G \backslash C_G(x)|$. Since $x \in G \backslash Z(G)$, $|G| \geqslant 2|C_G(x)|$. It follows that $d(x) > (|G| - |Z(G)|)/2$. Therefore by Dirac's theorem [12, p. 54], Γ_G is Hamiltonian. □

Proposition 2.3. *Let G be a non-abelian group. Then Γ_G is planar if and only if G is isomorphic to one of the groups S_3, D_8 or Q_8.*

Proof. It is easy to see that the non-commuting graphs of S_3, D_8 and Q_8 are all planar. Now suppose that Γ_G is planar. Since the complete graph of order 5 is not planar, we have $\omega(\Gamma_G) < 5$. Thus $G/Z(G)$ is a finite group, by the main result of [28]. Now we prove that $|Z(G)| \leqslant 5$. Suppose, for a contradiction, that $|Z(G)| > 5$ and consider a finite subset Z of $Z(G)$ with $|Z| > 5$. Since G is not abelian, there exist elements $x, y \in G$ such that $xy \neq yx$. Let $T = Zx \cup Zy$. Now the induced subgraph Γ_0 of Γ_G by T is a planar graph. Since Γ_0 is a finite planar graph, there exists a vertex $v \in T$ such that $d_{\Gamma_0}(v) \leqslant 5$ (see, e.g., [12, Corollary 3.5.9]). But for every two elements $z_1, z_2 \in Z$ we have xz_1 does not commute with yz_2. Thus $d_{\Gamma_0}(w) = |Z| > 5$ for all $w \in T$, which is a contradiction. Hence G is finite and so Γ_G is a finite graph. Thus there exists a vertex $x \in G \backslash Z(G)$ such that $d(x) \leqslant 5$ (see [12, Corollary 3.5.9]). Therefore $d(x) = |G \backslash C_G(x)| \leqslant 5$. It follows that $|C_G(x)| \leqslant |G|/2$ and we find $|G| \leqslant 10$. It is easy to see that there is only one non-abelian group of order 10 and its non-commuting graph is not planar. On the other hand, every non-abelian group of order less than 9 is isomorphic to one of the groups S_3, D_8 or Q_8. □

Proposition 2.4. *Let G be a non-abelian group and let S be a cut set of Γ_G. If x and y two vertices of $\Gamma_G \backslash S$ belong to distinct connected components, then S is a union of double cosets of $C_G(x) \cap C_G(y)$. In particular, if G is finite, then $\kappa(\Gamma_G) = t|Z(G)|$, where $t > 1$ is an integer.*

Proof. Let $H = C_G(x) \cap C_G(y)$ and $a \in G$ such that $HaH \cap S \neq \emptyset$. Then $HaH \subseteq S$, for if there exist elements $h_1, h_2 \in H$ such that $h_1ah_2 \notin S$, then $\{x, h_1ah_2\}$ and $\{y, h_1ah_2\}$ are edges of Γ_G, a contradiction. Now the first part follows, since the double cosets of H form a partition for G.

Suppose that $|S| = \kappa(\Gamma_G)$. Since S is a union of double cosets of H, it is a union of cosets of $Z(G)$. It follows that $\kappa(\Gamma_G) = t|Z(G)|$ for some integer $t \geqslant 1$. If $t = 1$, then $S = bZ(G)$ for some non-central element b in G. Let r and s two elements belong to distinct connected components of $\Gamma_G \backslash S$. Since $rs = sr$ and rb is adjacent to both r and s, we get a contradiction. This completes the proof. □

Remark 2.5. Let G be a group. Then for every maximal independent set S, $S \cup Z(G)$ is a maximal abelian subgroup of G. To see this if $x \in S$, then clearly $x^{-1} \in S$. Now suppose that $x, y \in S \cup Z(G)$, we want to show that $xy \in S \cup Z(G)$. Since xy commutes with each element of S and S is a maximal independent set, we have $xy \in S \cup Z(G)$.

Proposition 2.6. *Let G be a finite non-abelian group such that Γ_G is a regular graph. Then G is nilpotent of class at most* 3 *and $G = P \times A$, where A is an abelian group and P is a p-group* (p *is a prime*) *and furthermore Γ_P is a regular graph.*

Proof. Since $d(x) = |G| - |C_G(x)|$ for any vertex x, $|C_G(x)| = |C_G(y)|$ for any two non-central elements x, y. It follows that the conjugacy classes of G have only two sizes. Now Theorem 1 of [22] implies that G is nilpotent and it is a direct product of a non-abelian p-subgroup P and an abelian subgroup A, where p is prime and Γ_P is regular. A result of [21] yields that the nilpotence class of G is at most 3. □

Proposition 2.7. *Let G be a finite non-abelian group such that $|\{d(v) \mid v \in V(\Gamma_G)\}| = 2$. Then G is solvable.*

Proof. Since $d(v) = |G| - |C_G(v)|$ for each $v \in V(\Gamma_G)$, we conclude that the conjugacy classes of G are of 3 different sizes. Now it follows from a result of [23] that G is solvable. □

Remark 2.8. The non-commuting graph of S_3 and $\mathrm{GL}(2, q)$ $(q > 2)$ have two or three kinds of degrees respectively (see Proposition 3.26), but S_3 is not nilpotent and $\mathrm{GL}(2, q)$ is not solvable. So these last two propositions cannot be improved.

Paul Erdös, who was the first to consider the non-commuting graph of a group, posed the following problem in 1975 [27]: Let G be a group whose non-commuting graph has no infinite clique. Is it true that the clique number of Γ_G is finite? B.H. Neumann [27] answered positively Erdös' question as following.

Theorem 2.9. (B.H. Neumann [27]) *Let G be a group whose non-commuting graph has no infinite clique. Then $|G : Z(G)|$ is finite and in particular the clique number of Γ_G, $\omega(\Gamma_G)$ is finite.*

The dual question of Erdös may be posed as the following.

Question 2.10. *Let G be a group whose non-commuting graph has no infinite independent sets. Is it true that the independence number of G, $\alpha(\Gamma_G)$ is finite*?

We answer this question positively in some cases.

Theorem 2.11. *Let G be a group whose non-commuting graph has no infinite independent sets. If G is an Engel, locally finite, locally solvable or a linear group or a* 2*-group, then G is a finite group. In particular $\alpha(\Gamma_G)$ is finite.*

Proof. The hypothesis is equivalent to the following: Every abelian subgroup of G is finite. In particular, every cyclic subgroup of G is finite, so G is periodic. We know that an infinite locally finite group or an infinite 2-group contains an infinite abelian subgroup (see [37, Theorem 14.3.7]), and also every periodic locally solvable or linear group is locally finite. Thus in these cases G is finite. Now if G is an Engel group, since every abelian subgroup of G is finite, a result of Plotkin [36, Corollary, p. 55], implies that G is a finite nilpotent group. This completes the proof. □

Proposition 2.12. *Let G be a non-abelian group.*

(1) *If $\{x\}$ is a dominating set for Γ_G, then $Z(G)=1$, $x^2=1$ and $C_G(x)=\langle x\rangle$.*
(2) *If G is periodic, then $\gamma(\Gamma_G)=1$ if and only if G contains a normal abelian subgroup A with no element of order 2 and an element x of order 2 such that $a^x=a^{-1}$ for all $a\in A$ and $G=A\langle x\rangle$ and $A\cap\langle x\rangle=1$.*

Proof. (1) If $Z(G)$ contains a non-trivial element z, then xz is not adjacent to x, a contradiction, so $Z(G)=1$. Also if $x^2\neq 1$, then x^{-1} is not adjacent to x. Now since $Z(G)=1$ and x is adjacent to all vertices of Γ_G, $C_G(x)=\langle x\rangle$.

(2) Suppose that $\gamma(\Gamma_G)=1$. Thus G contains a non-central element x such that $\{x\}$ is a dominating set for Γ_G. By part (1) we have $C_G(x)=\langle x\rangle$ and $x^2=1$. It follows that $\langle x\rangle\cap\langle x\rangle^g=1$ for all $g\in G\backslash\langle x\rangle$. Now by [32, Theorem 5], $A=G\backslash\{x^g \mid g\in G\}$ is a normal abelian subgroup of G, such that $G=A\langle x\rangle$ and obviously $A\cap\langle x\rangle=1$. Thus G is a solvable periodic group which implies that G is locally finite. Let $a\in A$ be a non-trivial element of A, then $B=\langle a,a^x\rangle$ is a finite abelian normal subgroup of G. Hence x induces a fixed-point-free automorphism of order 2 in B, which implies that B is an abelian group of odd order (see, for example, [37, Exercise 10.5.1]) and also we have $b^x=b^{-1}$ for all $b\in B$.

Now assume that G contains a normal abelian subgroup A and an element x of order 2 with the properties stated in the proposition. It is easy to see that $Z(G)=1$ and none of the non-trivial elements of G commutes with x, so $\{x\}$ is a dominating set. □

Remark 2.13. Let G be a non-abelian group. A subset S of $V(\Gamma_G)$ is a dominating set if and only if $C_G(S)\subseteq Z(G)\cup S$. To see this, suppose that S is a dominating set. If $a\notin Z(G)\cup S$ then, by the definition of dominating set, there exists an element $x\in S$ such that $ax\neq xa$. Thus $a\notin C_G(S)$. It follows that $C_G(S)\subseteq S\cup Z(G)$.
Now assume that $C_G(S)\subseteq Z(G)\cup S$. If $a\notin Z(G)\cup S$, then by hypothesis, $a\notin C_G(S)$. Therefore a is adjacent to at least one element of S. This completes the proof.

Proposition 2.14. *Let G be a non-abelian group and X be a generating set for G. Then $X\backslash Z(G)$ is a dominating set for Γ_G. In particular, every non-abelian group G contains a dominating set generating a non-abelian subgroup.*

Proof. Suppose that $G=\langle X\rangle$, since G is non-abelian, $Y=X\backslash Z(G)\neq\emptyset$. Since G is generated by X, $C_G(Y)=Z(G)$. Now Remark 2.13 implies that Y is a dominating set for Γ_G. It is clear that for every generating set X of a non-abelian group, $\langle X\backslash Z(G)\rangle$ is a non-abelian group. Now the first part completes the proof. □

Remark 2.15. Proposition 2.14 shows that being finitely generated implies the existence of a finite dominating set. But the converse is not true, as there are groups G with $\gamma(\Gamma_G) = 1$, which are not finitely generated. There are groups whose non-commuting graphs have finite domination number, but the non-commuting graphs of some subgroups do not have a finite dominating set; for example, consider the Cartesian product of infinitely many copies of S_3 and the direct product of countably infinite copies of S_3, where the domination number of the former group is 2.

Another question which naturally arises is the finiteness of the domination number of the non-commuting graph of a normal subgroup of finite index in a group whose non-commuting graph has finite domination number. This is not true in general. For example, take a finite non-abelian group K with a fixed-point-free automorphism α (see, for example, [37, pp. 307–308]). Then let $H = \mathrm{Dr}_{i\in\mathbb{N}} K$ be the direct product of infinitely many copies of K. Now we define the automorphism $\bar{\alpha}$ of H by $\bar{\alpha}(\{x_i\}_{i\in\mathbb{N}}) = \{\alpha(x_i)\}_{i\in\mathbb{N}}$. Finally consider G as the semidirect product of H and $\langle\bar{\alpha}\rangle$. The non-trivial elements of $\langle\bar{\alpha}\rangle$ form a dominating set for Γ_G, and obviously Γ_H has no finite dominating set and the index of H in G is equal to the order of α which is finite.

Theorem 2.16. *Let G be a non-abelian group with a subgroup H of finite index. If $\gamma(\Gamma_H)$ is finite, then $\gamma(\Gamma_G)$ is finite (if H is an abelian group we define $\gamma(\Gamma_H) = 0$).*

Proof. We consider two cases:

Case 1. H is a non-abelian group. First assume that H is a maximal subgroup of G. Suppose that H is not normal in G. Let $H_G = \bigcap_{g\in G} H^g$. Then $\frac{G}{H_G}$ is a transitive permutation group on the left cosets of H and the stabilizer of H is H/H_G. Since H is a maximal subgroup which is not normal in G, H/H_G is a self-normalizing subgroup of G/H_G. Now [16, Theorem 4.2A(iv), p. 109] implies that $Z(\frac{G}{H_G}) = 1$. Let X be a finite subset of G such that $\frac{G}{H_G} = \{xH_G \mid x \in X\}$ and Y be a finite dominating set for Γ_H. We prove that $S = (X \cup Y)\backslash Z(G)$ is a dominating set for Γ_G. If $g \in C_G(S)\backslash S$, then since $Z(\frac{G}{H_G}) = 1$, we conclude that $g \in H_G$. Now since $g \in H\backslash Y$, we have $g \in Z(H)$. It follows that $g \in C_G(X) \cap Z(H)$ and so $g \in Z(G)$. It follows from Remark 2.13 that S is a dominating set for Γ_G.

Now assume that H is normal in G. Then, since we are assuming that H is a maximal subgroup of G, $|G : H| = p$ is a prime number. Let Y be a finite dominating set for Γ_H. If $Z(G) = C_G(Y)$, then Y is also a dominating set for Γ_G. Thus we may assume $C_G(Y)\backslash Z(G) \neq \emptyset$. If there exists an element $g \in C_G(Y)\backslash(H \cup Z(G))$, then $G = \langle g\rangle H$. Let $F = \{i \mid g^i \notin Z(G)$, for some i, $1 \leqslant i \leqslant p-1\}$. If $i \in F$, then $H \nleqslant C_G(g^i)$, since $G = \langle g^i\rangle H$. In this case let h_i be an element of H such $[h_i, g^i] \neq 1$ and let $K = \{h_i \mid i \in F\}$. Now we prove that the finite set $T := (K \cup \{g\} \cup (\bigcup_{i\in F} g^i Y) \cup Y)\backslash Z(G)$ is a dominating set for Γ_G. Let $x \in C_G(T)\backslash T$. If $x \in H$, then $x \in Z(H)$, since $x \in C_G(Y)$ and Y is a dominating set for Γ_H. Now, since $g \in T$, $x \in C_G(g) \cap Z(H) \leqslant Z(G)$. Thus in this case $x \in Z(G)$. Thus assume that $x = g^j h$ for some $j \in \{1, \dots, p-1\}$ and $h \in H$. Suppose that $h \notin Z(H)$, then since $g^j h \notin \bigcup_{i\in F} g^i Y$, $h \notin Y$, and so there exists an element $y \in Y$ such that $yh \neq hy$, a contradiction. Thus $[y, x] = [y, g^j h] \neq 1$. Thus $h \in Z(H)$. Therefore $h \in C_G(g) \cap Z(H) \leqslant Z(G)$. If $g^j \notin Z(G)$, then $[x, h_j] = [g^j h, h_j] \neq 1$, a contradiction. Thus $g^j \in Z(G)$ and so $x = g^j h \in Z(G)$, as required.

Now assume that $C_G(Y)\backslash Z(G) \subseteq H$. Then we prove that $Y \cup \{a\}$ is a dominating set for Γ_G, where a is any element in $G\backslash H$. Let $x \in C_G(Y \cup \{a\})\backslash(Y \cup \{a\})$. Then $x \in H \cup Z(G)$, by the inclusion $C_G(Y)\backslash Z(G) \subseteq H$. If $x \in H$, since Y is a dominating set for Γ_H, we have $x \in Z(H) \cap C_G(a) \leqslant Z(G)$. Now Remark 2.13 completes the proof.

Now we return to the case where H is not necessarily a maximal subgroup of G and argue by induction on $|G : H| = n$. If $n = 1$, there is noting to prove. Suppose, inductively, that $n > 1$. If H is a maximal subgroup of G, then by the previous case, we are done. Thus assume that H is not a maximal subgroup of G. In this case, there is a proper subgroup L of G such that $H \lneqq L$. Now since $|L : H| < n$, the induction implies that $\gamma(\Gamma_L)$ is finite, and again since $|G : L| < n$, by the induction we conclude that $\gamma(\Gamma_G)$ is finite, as required.

Case 2. Assume that H is an abelian group. We apply by induction on $|G : H| = n$. If there is a subgroup L of G such that $H \lneqq L \lneqq G$, then by induction hypothesis $\gamma(\Gamma_L)$ is finite. If L is an abelian group, then by induction $\gamma(\Gamma_G)$ is finite. If L is a non-abelian group, then by the Case 1 we have $\gamma(\Gamma_G)$ is finite. Thus we may assume that H is a maximal subgroup. Let $G = \bigcup_{i=1}^{n} Ha_i$. Since H is a maximal subgroup of G which is abelian and G is not abelian, $H \nleqslant Z(G)$. Now let $a \in H \backslash Z(G)$. Then we have $C_G(a) = H$ and we conclude that $Z(G) = \bigcap_{i=1}^{n} C_G(a_i) \cap C_G(a)$ and $\{a, a_1, \ldots, a_n\} \backslash Z(G)$ is a dominating set for Γ_G and the proof is complete. □

Corollary 2.17. *Suppose that G is a group and H is a subgroup of finite index in G. If there are some elements $x_1, \ldots, x_m \in H$ such that $Z(H) = \bigcap_{i=1}^{m} C_H(x_i)$, then there are $y_1, \ldots, y_n \in G$ such that $Z(G) = \bigcap_{i=1}^{n} C_G(y_i)$.*

Proposition 2.18.

(1) *The domination number of the non-commuting graph of every finite non-abelian simple group is* 2.
(2) *Let n be an integer and F be a field. If $n > 2$ or $n = 2$ and $|F| > 3$, then $\gamma(\Gamma_{\mathrm{GL}(n,F)}) = \gamma(\Gamma_{\mathrm{SL}(n,F)}) = 2$.*

Proof. (1) Let G be a finite non-abelian simple group. By Theorem B of [8], G is 2-generated. Let $X = \{a, b\}$ be a generating set for G, since G is a non-abelian simple group, $Z(G) = 1$ and so $X \cap Z(G) = \emptyset$. Thus Proposition 2.14 implies that $\gamma(\Gamma_G) \leqslant 2$. On the other hand, since G is simple, it does not contain a non-trivial proper normal subgroup, and so Proposition 2.12 yields that $\gamma(\Gamma_G) > 1$. Hence $\gamma(\Gamma_G) = 2$.

(2) Let $H = \mathrm{GL}(n, F)$ and $K = \mathrm{SL}(n, F)$. If F is finite, then the proof easily follows from part (1) and Proposition 2.12, since $K/Z(K)$ is a finite non-abelian simple group and $Z(H) \neq 1$. Now assume that F is an infinite field. Let X be the upper triangular matrix such that all its non-zero entries are 1 and let Y be the transpose of X. Then it is easy to check that $\{X, Y\}$ is a dominating set for both Γ_H and Γ_K. Thus $\gamma(\Gamma_H)$ and $\gamma(\Gamma_K)$ are at most 2. If $\gamma(\Gamma_K) = 1$, then there exists a matrix $A \notin \{I, -I\}$ such that $A^2 = I$ and $\{A\}$ is a dominating set, by Proposition 2.12. If $\mathrm{Char}(F) \neq 2$, then there exists a matrix $P \in H$ such that $P^{-1}AP$ has the form $B = \begin{bmatrix} I_{n_1} & 0 \\ 0 & -I_{n_2} \end{bmatrix}$, where n_1, n_2 are positive integers. Clearly there exists a non-scalar matrix $C \in H$ such that $BC = CB$ and $B \neq C$. Thus in this case $\gamma(\Gamma_K) = 2$. Now assume that $\mathrm{Char}(F) = 2$ and $\gamma(\Gamma_K) = 1$. There exists a matrix $Q \in H$ such that $Q^{-1}AQ$ is an upper triangular matrix whose all diagonal elements are 1. But we know that the group of all triangular matrices whose diagonal elements are 1, is a nilpotent group and so its center is non-trivial which contradicts Proposition 2.12. The proof for other case is similar. □

3. Groups with the same non-commuting graphs

In this section we consider the non-abelian groups with isomorphic non-commuting graphs. Note that if G and H are two groups, then $\Gamma_G \cong \Gamma_H$ if and only if there exists a bijective map $\phi: V(\Gamma_G) \to V(\Gamma_H)$ such that for every two distinct elements $x, y \in V(\Gamma_G)$, we have $xy = yx$ if and only if $\phi(x)\phi(y) = \phi(y)\phi(x)$.

The following lemma will be useful in the sequel.

Lemma 3.1. *Let G be a finite non-abelian group. If H is a group such that $\Gamma_G \cong \Gamma_H$, then H is a finite non-abelian group such that $|Z(H)|$ divides*

$$\gcd\bigl(|G| - \bigl|Z(G)\bigr|, |G| - \bigl|C_G(x)\bigr|, \bigl|C_G(x)\bigr| - \bigl|Z(G)\bigr| \colon x \in G\backslash Z(G)\bigr).$$

Proof. Since $\Gamma_G \cong \Gamma_H$, $0 \neq |G\backslash Z(G)| = |H\backslash Z(H)|$. Thus H is not abelian. Now $Z(H)$ cannot be infinite, since $|Z(H)| \leqslant |H\backslash Z(H)|$. Therefore H is a finite group, since $|H| = |H\backslash Z(H)| + |Z(H)|$. Now since $\Gamma_G \cong \Gamma_H$, $\{d(v) \mid v \in V(\Gamma_G)\} = \{d(v) \mid v \in V(\Gamma_H)\}$. But $d(v) = |G| - |C_G(v)|$ for any $v \in V(\Gamma_G)$, and clearly $|Z(H)|$ divides $d(v)$ for all $v \in V(\Gamma_H)$; since $|G| - |Z(G)| = |H| - |Z(H)|$, we have $|Z(H)|$ divides $|G| - |Z(G)|$. On the other hand, $|C_H(x)| - |Z(H)| = |C_G(\phi(x))| - |Z(G)|$ for all $x \in H\backslash Z(H)$, where $\phi: \Gamma_H \to \Gamma_G$ is a graph isomorphism. Thus $|Z(H)|$ divides $|C_G(x)| - |Z(G)|$ for all $x \in G\backslash Z(G)$. These imply that $|Z(H)|$ divides

$$\gcd\bigl(|G| - \bigl|Z(G)\bigr|, |G| - \bigl|C_G(x)\bigr|, \bigl|C_G(x)\bigr| - \bigl|Z(G)\bigr| \colon x \in G\backslash Z(G)\bigr). \qquad \square$$

Proposition 3.2. *Let G be a non-abelian group such that $\Gamma_G \cong \Gamma_{S_3}$. Then $G \cong S_3$.*

Proof. It follows from Proposition 2.3 that the only finite non-abelian group whose non-commuting graph is planar with 5 vertices, is S_3, so $G \cong S_3$. $\square$

The following result will reduce Conjecture 1.1 to *AC*-groups where a group is called an *AC-group* if the centralizer of every non-central element is abelian. The key in the proof is that if ϕ is a graph isomorphism between two non-commuting graphs Γ_G and Γ_H of two groups G and H, then the set of non-central elements of centralizer of an element in G is mapped to the set of non-central elements of centralizer of an element in H, by ϕ, and moreover, their non-commuting graphs of these centralizers are isomorphic. The difficulty that has arisen in the proving Conjecture 1.1 is that if the non-commuting graph of a group is null, then we only know that the group is abelian, and no other information can be retrieved from the non-commuting graph.

Proposition 3.3. *Let $n > 2$ be an integer such that every two finite non-abelian AC-groups with isomorphic non-commuting graphs and the clique number less than or equal to n have the same size. Then for every two finite non-abelian groups G and H if $\Gamma_G \cong \Gamma_H$ and $\omega(\Gamma_G) \leqslant n$, we have $|G| = |H|$.*

Proof. We apply induction on $|G|$. Since G is non-abelian, $|G| \geqslant 6$. If $|G| = 6$ then $G \cong S_3$ and Proposition 3.2 implies that $G \cong H$, in particular $|G| = |H|$. Now suppose, inductively that, $|G| > 6$ and if G_1 and H are two finite non-abelian groups such that $\Gamma_{G_1} \cong \Gamma_H$ with the clique number less than or equal to n and $|G_1| < |G|$, then $|G_1| = |H|$. Note that for every $x \in G \backslash Z(G)$, $|C_G(x) \backslash Z(C_G(x))| = |C_H(\phi(x)) \backslash Z(C_H(\phi(x))|$, where $\phi : \Gamma_G \to \Gamma_H$ is a graph isomorphism. Also if $C_G(x)$ is not abelian then $\Gamma_{C_G(x)} \cong \Gamma_{C_H(\phi(x))}$. Thus G is an AC-group if and only if H is an AC-group. Also it is clear that the clique number of the non-commuting graph of every subgroup of G, in particular $C_G(x)$, is less than or equal to n. Hence by the hypothesis we may assume that G is not an AC-group. So there exists an element $a \in G \backslash Z(G)$ such that $C_G(a)$ is not abelian and $\Gamma_{C_G(a)} \cong \Gamma_{C_H(\phi(a))}$. Since a is non-central, $|C_G(a)| < |G|$ and so by induction hypothesis $|C_G(a)| = |C_H(\phi(a))|$. On the other hand, since $\Gamma_G \cong \Gamma_H$, $|C_G(a) \backslash Z(G)| = |C_H(\phi(a)) \backslash Z(H)|$. It follows that $|Z(G)| = |Z(H)|$ which implies $|G| = |H|$. $\square$

Remark 3.4. It is not hard to prove that the requested property for the AC-groups in Proposition 3.3 is true, when $n \leqslant 4$, since one can easily find (with help of well-known results) the structure of $G/Z(G)$ for AC-groups with $n = \omega(\Gamma_G) \leqslant 4$. It is enough to note that such groups are a union of n centralizers whose intersection is $Z(G)$ and their union is 'irredundant.'

Proposition 3.3 shows the importance of AC-groups for proving Conjecture 1.1. The AC-groups have been extensively studied by many authors (see, e.g., [22,29,38]) and there is a complete classification of finite non-nilpotent AC-groups by R. Schmidt (see [30]). We start with some examples of AC-groups.

Lemma 3.5.

(1) *For any field F, the group* $\mathrm{GL}(2, F)$ *is an AC-group.*
(2) *Every free group is an AC-group.*
(3) *Every free solvable group is an AC-group.*

Proof. (1) Let a be a non-central element of $G = \mathrm{GL}(2, F)$. Then the minimal polynomial a in F is of degree 2. Thus a is a cyclic matrix. It follows easily that the centralizer of a in the ring of all 2×2 matrices over F, $\mathrm{M}_n(F)$, is equal to the algebra generated by a and F in $\mathrm{M}_n(F)$. Since the latter algebra is commutative and contains $C_G(a)$, the proof of (1) is complete.

(2) If K is a non-cyclic free group, then $C_G(x) = \langle x \rangle$ for every non-trivial element $x \in K$. This completes the proof of (2).

(3) This is the last corollary in [38]. $\square$

The following characterization of AC-groups may be useful in some points.

Lemma 3.6. *The following are equivalent on a group G.*

(a) *G is an AC-group.*
(b) *If $[x, y] = 1$, then $C_G(x) = C_G(y)$, whenever $x, y \in G \backslash Z(G)$.*
(c) *If $[x, y] = [x, z] = 1$, then $[y, z] = 1$, whenever $x \in G \backslash Z(G)$.*
(d) *If A and B are subgroups of G and $Z(G) \lneqq C_G(A) \leqslant C_G(B) \lneqq G$, then $C_G(A) = C_G(B)$.*

Proof. This is straightforward (see also [29, Lemma 3.2]). $\square$

Remark 3.7. By Lemma 3.6, if G is a non-abelian AC-group, then the commutativity relation on $G \backslash Z(G)$ is transitive and $\gamma(\Gamma_G) \leqslant 2$.

Let G be a group and $\mathfrak{C}(G) = \{C_G(X) \mid X \leqslant G\}$, where $C_G(X) = \bigcap_{x\in X} C_G(x)$. For $U, V \in \mathfrak{C}(G)$, define $U \wedge V = U \cap V$ and

$$U \vee V = \bigcap \bigl\{X \mid X \in \mathfrak{C}(G),\ U \leqslant X,\ V \leqslant X\bigr\}.$$

The set $\mathfrak{C}(G)$ with the operations $\wedge$ and $\vee$ is a lattice, where $Z(G)$ and G are 0 and 1 of this lattice, respectively. Following R. Schmidt [30], a group G is said to be an $\mathfrak{M}$-group if every maximal element of $\mathfrak{C}(G)$ is also a minimal element of $\mathfrak{C}(G)$. It is easy to see that a group G is an $\mathfrak{M}$-group if and only if G is an AC-group (see [30, Lemma 2.9]). The finite non-nilpotent $\mathfrak{M}$-groups (i.e. AC-groups) are completely classified by R. Schmidt (see [30, Satz 5.9 and Satz 5.12]).

Theorem 3.8. [30, Satz 5.9] *Let G be a finite non-solvable group. Then G is an AC-group if and only if G satisfies one of the following conditions*:

(1) $G/Z(G) \cong \mathrm{PSL}(2, p^n)$ *and* $G' \cong \mathrm{SL}(2, p^n)$, *where p is a prime and* $p^n > 3$.
(2) $G/Z(G) \cong \mathrm{PGL}(2, p^n)$ *and* $G' \cong \mathrm{SL}(2, p^n)$, *where p is a prime and* $p^n > 3$.
(3) $G/Z(G) \cong \mathrm{PSL}(2, 9)$ *and G' is isomorphic to*

$$\begin{aligned}\mathcal{A} \cong \bigl\langle c_1, c_2, c_3, c_4, k \mid\ & c_1^3 = c_2^2 = c_3^2 = c_4^2 = (c_1c_2)^3 = (c_1c_3)^2 = (c_2c_3)^3 = (c_3c_4)^3 = k^3,\\ & (c_1c_4)^2 = k,\ c_2c_4 = k^3c_4c_2,\ kc_i = c_ik\ (i = 1, \ldots, 4),\ k^6 = 1\bigr\rangle.\end{aligned}$$

(4) $G/Z(G) \cong \mathrm{PGL}(2, 9)$ *and* $G' \cong \mathcal{A}$.

Lemma 3.9. *In Theorem* 3.8, *if $p = 2$ or $|Z(G)|$ is odd or $G' \cap Z(G) = 1$, then* $G \cong Z(G) \times \mathrm{PSL}(2, 2^n)$.

Proof. If $G/Z(G) \cong \mathrm{PSL}(2, 9)$ or $\mathrm{PGL}(2, 9)$ and $G' \cong \mathcal{A}$, then it is easy to see that $G' \cap Z(G) = Z(G')$ and $Z(G')$ is a cyclic group of order 6 (note that $|\mathcal{A}| = 6|\mathrm{PSL}(2, 9)|$). Now suppose that $G/Z(G) \cong \mathrm{PSL}(2, p^n)$ or $\mathrm{PGL}(2, p^n)$ and $G' \cong \mathrm{SL}(2, p^n)$. We have $Z(G')$ is of order 1 (if $p = 2$) and of order 2 (if $p > 2$), since $G' \cong \mathrm{SL}(2, p^n)$. It follows that $G' \cap Z(G) = Z(G')$ is always of order 1 or 2 and if $|Z(G)|$ is odd, then $G' \cap Z(G) = 1$, also $p = 2$ if and only if $G' \cap Z(G) = 1$.

Now suppose that $G' \cap Z(G) = 1$ or equivalently $p = 2$.

Since $p = 2$, $\mathrm{PGL}(2, 2^n) = \mathrm{SL}(2, 2^n) \cong \mathrm{PSL}(2, 2^n)$. Thus we have $G/Z(G) \cong \mathrm{PSL}(2, 2^n)$. Therefore $G'Z(G) = G$, since $\mathrm{PSL}(2, p^n)$ is a non-abelian simple group. It follows that $G \cong Z(G) \times \mathrm{PSL}(2, 2^n)$. $\square$

Theorem 3.10. [30, Satz 5.12] *Let G be a finite non-abelian solvable group. Then G is an AC-group if and only if G satisfies one of the following properties*:

(1) *G is non-nilpotent and it has an abelian normal subgroup N of prime index and* $\omega(\Gamma_G) = |N : Z(G)| + 1$.
(2) *$G/Z(G)$ is a Frobenius group with Frobenius kernel and complement $F/Z(G)$ and $K/Z(G)$, respectively and F and K are abelian subgroups of G; and* $\omega(\Gamma_G) = |F : Z(G)| + 1$.

(3) *$G/Z(G)$ is a Frobenius group with Frobenius kernel and complement $F/Z(G)$ and $K/Z(G)$, respectively*; *and K is an abelian subgroup of G, $Z(F)=Z(G)$, and $F/Z(G)$ is of prime power order*; *and $\omega(\Gamma_G)=|F:Z(G)|+\omega(\Gamma_F)$.*
(4) *$G/Z(G)\cong S_4$ and V is a non-abelian subgroup of G such that $V/Z(G)$ is the Klein 4-group of $G/Z(G)$*; *and $\omega(\Gamma_G)=13$.*
(5) *$G=A\times P$, where A is an abelian subgroup and P is an AC-subgroup of prime power order.*

Lemma 3.11. *Let G be a non-abelian finite group such that $G/Z(G)\cong S_4$. Suppose that V is a non-abelian subgroup of G such that $V/Z(G)$ is the Klein 4-group of $G/Z(G)$. If $\Gamma_G\cong\Gamma_H$ for some group H. Then $|G|=|H|$.*

Proof. By the proof of part (3) of Satz 5.12 in [30], we have G is an AC-group and contains elements x, y, z such that $|C_G(x)|=4|Z(G)|$, $|C_G(y)|=3|Z(G)|$ and $|C_G(z)|=2|Z(G)|$. Now Lemma 3.1 implies that $|Z(H)|$ divides $|Z(G)|$. Thus there exists an integer m such that $|Z(G)|=m|Z(H)|$. On the other hand, since $|G|-|Z(G)|=|H|-|Z(H)|$, we have $23m+1=|H|/|Z(H)|$. Also there exists an element $a\in H$ such that $|C_G(x)|-|Z(G)|=|C_H(a)|-|Z(H)|$ and so $3m+1=|C_H(a)|/|Z(H)|$. Thus $3m+1$ divides $23m+1$ which yields that $3m+1$ divides $m+7$; but $3m+1>m+7$ for every $m>3$ and if $m=2$, then $3m+1\nmid m+7$. Thus $m=1$ or 3. By a similar argument, $m+1\mid 23m+1$. Therefore $m=1$ and $|Z(G)|=|Z(H)|$. Hence $|H|=|G|$. □

Lemma 3.12. *Let G be a non-abelian finite group such that $G/Z(G)$ be a Frobenius group with Frobenius kernel and complement $F/Z(G)$ and $K/Z(G)$, respectively and F and K are abelian subgroups of G. If $\Gamma_G\cong\Gamma_H$ for some group H, then $|G|=|H|$.*

Proof. Let $m=|Z(G)|$, $n=|Z(H)|$ and $|F|=lm$ and $|K|=sm$. Since $\Gamma_G\cong\Gamma_H$, $|G|-|Z(G)|=|H|-|Z(H)|$, $|K|-|Z(G)|=|C_H(g)|-|Z(H)|$ and $|F|-|Z(G)|=|C_H(h)|-|Z(H)|$, for some $g,h\in H$. It follows that $|H|/|Z(H)|=(ls-1)m/n+1$, $|C_H(h)|/|Z(H)|=(l-1)m/n+1$ and $n=|Z(H)|$ divides $\gcd(l-1,s-1)m$. Thus $(l-1)m/n+1$ divides $(ls-1)m/n+1$ and so $(l-1)m+n$ divides $(ls-1)m+n$. Since $G/Z(G)$ is a Frobenius group, $l-1=st$ for some integer t (see [19, Satz 8.3]). Therefore $\gcd(s-1,l-1)=\gcd(t,s-1)$ and so n divides tm which implies $tm\geqslant n$. Since $(ls-1)m+n=[(l-1)m+n]s-(s-1)(n-m)$, we have $(l-1)m+n$ divides $(s-1)(n-m)$. But $tm\geqslant m$ and $tm\geqslant n$, thus $tm\geqslant|n-m|$ and so

$$|n-m|(s-1)\leqslant(s-1)tm<stm=(l-1)m<(l-1)m+n.$$

Hence $|(n-m)(s-1)|<(l-1)m+n$ and so $(l-1)m+n$ divides $(n-m)(s-1)$ if and only if $m=n$. Therefore $|Z(G)|=|Z(H)|$ which implies that $|G|=|H|$, as required. □

Lemma 3.13. *Let G be a finite non-nilpotent AC-group containing a normal abelian subgroup N with prime index in G. If $\Gamma_G\cong\Gamma_H$ for some group H which is not nilpotent. Then $|G|=|H|$.*

Proof. Let $|G:N|=p$. By the proof of part (1) of Satz 5.12 of [30], we have $|C_G(x)|=p|Z(G)|$ or $C_G(x)=N$ for every non-central element $x\in G$. Since G is an AC-group, every maximal independent subset S of Γ_G is either equal to $N\setminus Z(G)$ or a subset of size $(p-1)|Z(G)|$ and the number of the independent subsets of the latter size is exactly $|N:Z(G)|$ (note that if

$|N\backslash Z(G)|=(p-1)|Z(G)|$, then by [30, Korollar 5.17], G is nilpotent, a contradiction); also $N\backslash Z(G)$ is the unique independent subset of size $|N\backslash Z(G)|$. Since $\Gamma_G\cong\Gamma_H$ and G is an AC-group, we have H is a finite non-abelian AC-group and

$$\left|\left\{d(v)\mid v\in V(\Gamma_G)\right\}\right|=\left|\left\{d(v)\mid v\in V(\Gamma_H)\right\}\right|=2.$$

Thus Proposition 2.7 implies that H is solvable. Now by Lemmas 3.11 and 3.12 and since H is not nilpotent, we may assume that H satisfies either property (1) or (3) in Theorem 3.10. If H satisfies (1), then H contains a normal abelian subgroup B of prime index q, and $\omega(\Gamma_H)=|B:Z(H)|+1$. Since $\Gamma_G\cong\Gamma_H$, we have

$$\left|B:Z(H)\right|=\left|N:Z(G)\right|. \tag{I}$$

On the other hand, as we mentioned for G, we have every maximal independent subset of Γ_H is either equal to $B\backslash Z(H)$ or a subset of size $(q-1)|Z(H)|$ and the number of the independent subsets of the size $(q-1)|Z(H)|$ is exactly $|B:Z(H)|$ and $B\backslash Z(H)$ is the unique independent subset of size $|B\backslash Z(H)|$. Therefore, since $\Gamma_G\cong\Gamma_H$, we must have

$$\left|N\backslash Z(G)\right|=\left|B\backslash Z(H)\right|. \tag{II}$$

Now it follows from (I) and (II), $|Z(G)|=|Z(H)|$. Hence, in this case we have $|G|=|H|$.

Now assume that H satisfies the property (3). Then $H/Z(H)$ is a Frobenius group with Frobenius kernel and complement $F/Z(H)$ and $K/Z(H)$, respectively; where F and K are subgroups of H with $Z(F)=Z(H)$, $F/Z(H)$ is a q-group (for some prime q), and K is abelian. Also $\omega(\Gamma_H)=|F:Z(H)|+\omega(\Gamma_F)$. Since $|F/Z(H)|>1$ and $Z(F)=Z(H)$, F is not abelian. So $\omega(\Gamma_F)\geqslant 3$. On the other hand, we have every maximal independent set of Γ_H is equal to either the non-central elements of a conjugate of K (whose size is the same integer $|K\backslash Z(H)|$) or a maximal independent subset of Γ_F. Note that the number of maximal independent subsets of Γ_F is at least 3. Also we note that the size of a maximal independent subset as $K^x\backslash Z(H)$ (for some $x\in H$) is not equal to one in Γ_F, since $|K/Z(H)|$ divides $|F/Z(H)|-1$ (see [19, Satz 8.3]). Now we define $\mathcal{S}_H:=\{S\mid S$ is a maximal independent subset in $\Gamma_H\}$, $L_H:=\{|S|\mid S\in\mathcal{S}_H\}$ and define $i_H:\mathcal{S}_H\to\mathbb{N}$ with $i_H(S)=|\{S'\in\mathcal{S}_H\mid |S|=|S'|\}|$ and $T_H:=\{i_H(S)\mid S\in\mathcal{S}_H\}$ and similarly define $\mathcal{S}_G$, L_G, i_G, T_G for G. Since $\Gamma_G\cong\Gamma_H$, $|L_H|=|L_G|=2$. Therefore there exist $S_1,S_2\in\mathcal{S}_H$ such that for each $S\in\mathcal{S}_H$ we have $|S|=|S_1|=|K\backslash Z(H)|$ or $|S|=|S_2|$, where $S\subseteq F\backslash Z(H)$. Now since $T_H=T_G$, from what we have discussed above, it follows that $i_H(S)=1$ for some $S\in\mathcal{S}_H$. But obviously, $i_H(S_1)>1$ and since F is not abelian $i_H(S_2)>1$, a contradiction. Hence H cannot satisfy the property (3) in Theorem 3.10. This completes the proof. □

Proposition 3.14. *Let H be a group such that $\Gamma_G\cong\Gamma_H$, where G is a finite non-solvable AC-group. Then $|G|=|H|$.*

Proof. First note that H is finite, by Lemma 3.1. By Theorem 3.8, we may distinguish the following cases:

Case (1). $G/Z(G)\cong\mathrm{PSL}(2,9)$ or $\mathrm{PGL}(2,9)$ and $G'\cong\mathcal{A}$, where $\mathcal{A}$ is defined in Theorem 3.8. We first assume that $G/Z(G)\cong\mathrm{PSL}(2,9)$. Then by the proof of [30, Lemma 5.3(b)], G contains two elements x, y such that $|\frac{C_G(x)}{Z(G)}| = 4$ and $|\frac{C_G(y)}{Z(G)}| = 5$. It follows that H contains two elements x' and y' such that $3|Z(G)| = |C_H(x')| - |Z(H)|$ and $4|Z(G)| = |C_H(y')| - |Z(H)|$. Thus we conclude that $|Z(H)|$ divides $|Z(G)|$. Thus $|Z(G)| = |Z(H)|k$ for some positive integer k. Thus $|\frac{C_H(y')}{Z(H)}| = 4k + 1$. Also $|G| - |Z(G)| = |H| - |Z(H)|$ and so $|\frac{H}{Z(H)}| = 359k + 1$. Therefore $4k + 1$ divides $359k + 1$, but $355k = 359k + 1 - 4k - 1$ and $\gcd(4k + 1, k) = 1$, so we have $4k + 1$ divides $355 = 5 \times 71$. Now it is easy to see that $k = 1$. Hence $|Z(G)| = |Z(H)|$ and so $|G| = |H|$.

Now assume that $\frac{G}{Z(G)} \cong \mathrm{PGL}(2, 9)$. By the proof of [30, Lemma 5.3(b)], G contains two elements r and s such that $|\frac{C_G(r)}{Z(G)}| = 8$ and $|\frac{C_G(s)}{Z(G)}| = 10$. It follows that H contains two elements r' and s' such that $7|Z(G)| = |C_H(r')| - |Z(H)|$ and $9|Z(G)| = |C_H(s')| - |Z(H)|$. Thus $|Z(H)|$ divides $|Z(G)|$. Suppose $|Z(G)| = |Z(H)|h$ for some positive integer h. It follows that $|\frac{C_H(r')}{Z(H)}| = 7h + 1$. On the other hand, since $|G| - |Z(G)| = |H| - |Z(H)|$, we have $|\frac{H}{Z(H)}| = 719h + 1$. Hence $7h + 1$ must divide $719h + 1$ and so $7h + 1$ divides $712h$. Since $\gcd(7h + 1, h) = 1$, $7h + 1$ divides $712 = 8 \times 89$. Now it is easy to see that $h = 1$. It follows that $|Z(G)| = |Z(H)|$ and so $|G| = |H|$.

Now we deal with the second case:

Case (2). $G/Z(G) \cong \mathrm{PSL}(2, p^n)$ or $\mathrm{PGL}(2, p^n)$ and in both cases $G' \cong \mathrm{SL}(2, p^n)$, where p is a prime, $p^n > 3$. Let $q = p^n$ and $k = \gcd(q - 1, 2)$. The group $\mathrm{PGL}(2, q)$ (respectively $\mathrm{PSL}(2, q)$) has a partition $\mathcal{P}$ consisting of $q + 1$ Sylow p-subgroups, $\frac{(q+1)q}{2}$ cyclic subgroups of order $q - 1$ (respectively $\frac{q-1}{k}$) and $\frac{(q-1)q^2}{2}$ cyclic subgroups of order $q + 1$ (respectively $\frac{q+1}{k}$) (see [19, pp. 185–187 and 193]). Now [30, (5.3.3) in p. 112]] states that if $x \in G\backslash Z(G)$, then $C_G(x)/Z(G)$ belongs to $\mathcal{P}$. Suppose that $G/Z(G) \cong \mathrm{PGL}(2, q)$ (respectively $\mathrm{PSL}(2, q)$). Thus there exist elements $g_1, g_2, g_3 \in G\backslash Z(G)$ such that $|C_G(g_1)|/|Z(G)| = q$, $|C_G(g_2)|/|Z(G)| = q - 1$ (respectively $\frac{q-1}{k}$), $|C_G(g_3)|/|Z(G)| = q + 1$ (respectively $\frac{q+1}{k}$). Since $\Gamma_G \cong \Gamma_H$, there exist elements $h_1, h_2, h_3 \in H\backslash Z(H)$ such that $|H| - |C_H(h_i)| = |G| - |C_G(g_i)|$ and $|C_H(h_i)| - |Z(H)| = |C_G(g_i)| - |Z(G)|$ for $i = 1, 2, 3$. Now it follows from Lemma 3.1 that $|Z(H)|$ divides $|Z(G)| \gcd(q - 2, q - 1, q)$ (respectively $|Z(G)| \gcd(\frac{q-1}{k} - 1, \frac{q+1}{k} - 1, q - 1)$). Therefore $|Z(G)| = |Z(H)|m$ for some positive integer m.

Thus so far we have proved that if G is a finite non-solvable AC-group and $\Gamma_G \cong \Gamma_K$ for some group K, then $|Z(K)|$ divides $|Z(G)|$. Since $\Gamma_G \cong \Gamma_H$ and G is an AC-group, H is so. Therefore if H is also non-solvable, we have $Z(G)$ divides $Z(H)$ and so $|Z(G)| = |Z(H)|$ which implies $|G| = |H|$, as required. Hence we may assume that H is solvable. By Theorem 3.10, H satisfies one of the properties (1) to (5) and since we want to prove $|G| = |H|$, by Lemmas 3.11–3.13, we may assume that H satisfies either the property (3) or (5). Now we prove that H does not satisfy the property (5), since otherwise, $|F/Z(H)|$ is a prime power. Now it is easy to check that, in any case, there exists $j \in \{1, 2, 3\}$ such that $|\frac{C_H(h_j)}{Z(H)}|$ is relatively prime to $|\frac{C_H(h_i)}{Z(H)}|$ for $i \in \{1, 2, 3\}\backslash\{j\}$. But since these integers must divide $|H/Z(H)|$, we get a contradiction.

Thus it remains only to deal with the case where H satisfies the property (3). In this case H contains a normal subgroup F and an abelian subgroup K such that $H/Z(H)$ is a Frobenius group with kernel $F/Z(H)$ and complement $K/Z(H)$ also $F/Z(H)$ is a group of prime power order r^t (for some prime r) and $Z(F) = Z(H)$. Note that since $H/Z(H)$ is a Frobenius group, $|K/Z(H)|$ is relatively prime to $|F/Z(H)|$ and for each $h \in H \backslash Z(H)$, we have

$$C_H(h) \leqslant F \quad \text{or} \quad C_H(h) = K^g \quad \text{for some } g \in H. \tag{$*$}$$

Let $a_i := |\frac{C_H(h_i)}{Z(H)}|$ for each $i \in \{1,2,3\}$. Then in any case ($\frac{G}{Z(G)} \cong \mathrm{PSL}(2,q)$ or $\frac{G}{Z(G)} \cong \mathrm{PGL}(2,q)$) we have $a_1 = (q-1)m+1$. Also we find that $|\frac{H}{Z(H)}|$ is either $(q^3-q-1)m+1$ or $(q\frac{q^2-1}{k}-1)m+1$.

Now assume that $\frac{G}{Z(G)} \cong \mathrm{PSL}(2,q)$ and $k = \gcd(q-1,2) = 2$. Then $a_2 = \frac{q-3}{2}m+1$ and $a_3 = \frac{q-1}{2}m+1$. Since $\gcd(a_3,a_i) = 1$ for $i \in \{1,2\}$ and $|F/Z(H)|$ is a prime power, we have $a_3 = |K/Z(H)|$, a_1 and a_2 are r-powers. Since $a_1 > a_2$ and both are r-powers, we have $\gcd(a_1,a_2) = a_2$ and so a_2 must divide $a_1 - a_2 = \frac{q+1}{2}m$. Now since $\gcd(a_2,m) = 1$, a_2 divides $\frac{q+1}{2}$. It follows that $m \in \{1,2\}$ and $q = 5$ (note that $q > 3$ and since $k = 2$, q is odd, so $q \geqslant 5$). Thus $m = 2$ and $q = 5$. Hence $|\frac{H}{Z(H)}| = (\frac{q(q^2-1)}{2} - 1)m + 1 = 119$ and $a_1 = 9$ which must divide 119, a contradiction.

Now suppose that $\frac{G}{Z(G)} \cong \mathrm{PSL}(2,q)$ and $k = \gcd(q-1,2) = 1$ (or $\frac{G}{Z(G)} \cong \mathrm{PGL}(2,q)$). It follows that $a_2 = (q-2)m+1$ and $a_3 = qm+1$. Since, in this case, $\gcd(a_1,a_i) = 1$ for $i \in \{2,3\}$ and $|F/Z(H)|$ is a prime power, we have $a_1 = |K/Z(H)|$, a_2 and a_3 are r-powers. Since $a_3 > a_2$ and these integers are r-powers, $\gcd(a_3,a_2) = a_2$ and a_2 must divide $a_3 - a_2 = 2m$. Therefore a_2 divides 2, since $\gcd(m,a_2) = 1$. But this is impossible, since $a_2 = (q-2)m+1 > 2$. This completes the proof. □

Remark 3.15. According to Propositions 3.3 and 3.14, the problem of whether two finite non-abelian groups with the same non-commuting graphs have the same size, is equivalent to that of whether two finite non-abelian solvable AC-groups with the same non-commuting graphs have the same size.

In the following we give certain groups for which Conjecture 1.1 is true.

Theorem 3.16. *If G is a group and $n > 2$ is an integer, then the following hold*:

(1) $\varGamma_G \cong \varGamma_{S_n}$, *then* $|G| = |S_n|$.
(2) *If* $n > 3$ *and* $\varGamma_G \cong \varGamma_{A_n}$, *then* $|G| = |A_n|$.

Proof. (1) Let a and b be the cycles $(1,2,\dots,n)$ and $(1,2,\dots,n-1)$, respectively. Then $C_{S_n}(a) \cong \langle a \rangle$ and $C_{S_n}(b) \cong \langle b \rangle$. Now Lemma 3.1 implies that G is a finite group and $|Z(G)|$ divides $n-(n-1) = 1$. Thus $|Z(G)| = 1$ and so $|G| = |S_n|$.

(2) Suppose first that n is odd. Then $a \in A_n$ and $C_{A_n}(a) \cong \langle a \rangle$. By Lemma 3.1, G is a finite group and $|Z(G)|$ divides $\gcd(n-1, \frac{n!}{2} - 1) = 1$, since $n > 3$. Hence $Z(G) = 1$ and so, in this case, $|G| = |A_n|$.

Now assume that n is even. Then $b \in A_n$ and $C_{A_n}(b) \cong \langle b \rangle$. It follows from Lemma 3.1, that $|Z(G)|$ divides $\gcd(n-2, \frac{n!}{2} - 1) = 1$, since $n > 3$. Therefore $Z(G) = 1$ and so $|G| = |A_n|$. This completes the proof. □

Proposition 3.17. *Let $k > 0$ be an integer and $G = \mathrm{GL}(2^k, 2)$. If H is a group such that $\Gamma_G \cong \Gamma_H$, then $|G| = |H|$.*

Proof. We apply by induction on k. The case $k = 1$ follows from Proposition 3.2. Assume that $k > 1$. Let S be the set of all matrices of the form

$$\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix},$$

where $A \in \mathrm{GL}(2^{k-1}, 2)$ and I is the identity matrix in $\mathrm{GL}(2^{k-1}, 2)$. It is easy to see that $C_G(S)$ consists of all matrices as the following form

$$\begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix},$$

where $B \in \mathrm{GL}(2^{k-1}, 2)$. Thus $C_G(S) \cong \mathrm{GL}(2^{k-1}, 2)$. Now suppose that $\phi : \Gamma_G \to \Gamma_H$ is a graph isomorphism. Then $\Gamma_{C_G(S)} \cong \Gamma_{C_H(\phi(S))}$, so by induction hypothesis we have $|C_G(S)| = |C_H(\phi(S))|$. Since $|C_G(S)\backslash Z(G)| = |C_H(\phi(S))\backslash Z(H)|$, it follows that $|Z(G)| = |Z(H)|$ and so $|G| = |H|$. □

Proposition 3.18. *Let G and H be two finite non-abelian groups such that $|G\backslash Z(G)| = |H\backslash Z(H)|$. If $|G| = p^n$ and $|H| = p^m$, where p is a prime number and n, m are positive integers, then $|G| = |H|$. In particular, if $\Gamma_G \cong \Gamma_H$, then $|G| = |H|$.*

Proof. We have $|Z(G)| = p^r$ and $|Z(H)| = p^s$ for some positive integers r, s. Therefore, by hypothesis, $p^n - p^r = p^m - p^s$. Clearly we have $s = r$. It follows that $|G| = |H|$. □

In the following we prove Conjecture 1.1 for dihedral groups.

Proposition 3.19. *Let $m > 2$ be an integer and $G = D_{2m}$. If $\Gamma_G \cong \Gamma_H$ for some group H, then $|G| = |H|$ and H contains an abelian subgroup of index 2.*

Proof. By Lemma 3.1, H is finite. If m is odd, then $Z(G) = 1$ and there is an element $b \in G$ such that $|C_G(b)| = 2$. Thus there exists an element $b' \in H\backslash Z(H)$ such that $2 - 1 = |C_H(b')| - |Z(H)|$. Thus $|C_H(b')| = |Z(H)| + 1$. It follows that $Z(H) = 1$ and so $|G| = |H|$.

If m is even, then $|Z(G)| = 2$ and there exist elements $b_1, b_2 \in G$ such that $|C_G(b_1)| = 4$ and $|C_G(b_2)| = m$. Thus $4 - 2 = |C_H(b_1')| - |Z(H)|$, for some $b_1' \in H\backslash Z(H)$. Thus we find $|Z(H)| \mid 2$. If $Z(H) = 1$, then $|H| = 2m - 2 + 1 = 2m - 1$ and $2m - m = 2m - 1 - |C_H(b_2')|$, for some $b_2' \in H\backslash Z(H)$. Therefore $|C_H(b_2')| = m - 1$. Therefore $m - 1$ divides $2m - 1$ which follows $m - 1$ divides 1. Hence $m = 2$ which is a contradiction. Hence $|Z(H)| = 2$ and so $|G| = |H|$.

Now since G has a cyclic subgroup A of order m (which is a maximal abelian subgroup), $\Gamma_G \cong \Gamma_H$ and $|G| = |H|$, Remark 2.5 implies that $B = \phi(A\backslash Z(G)) \cup Z(H)$ is an abelian subgroup of order m in H, where ϕ is a graph isomorphism from Γ_G to Γ_H. Clearly B is of index 2 in H, as required. □

Proposition 3.20. *Let G be a non-abelian group of order p^3 (p is a prime) and A be a finite abelian group. If $\Gamma_{A\times G} \cong \Gamma_H$ for some group H, then $|A \times G| = |H|$ and $H = Q \times B$, where B is an abelian subgroup and Q is a non-abelian p-group.*

Proof. By Lemma 3.1, H is finite. The center of $A \times G$ is of order $p|A|$ and the centralizer of every non-central element of $A \times G$ is of order $p^2|A|$. It follows that $\Gamma_{A\times G}$ is a regular graph, and so Proposition 2.6 implies that H is a direct product of a non-abelian q-subgroup Q (q is a prime and Γ_Q is regular) and an abelian subgroup B. Let $|Q| = q^s$, $|Z(Q)| = q^n$, $|B| = b$ and $|A| = a$. Note that $|C_Q(x_1)| = |C_Q(x_2)| = q^r$, for all non-central elements $x_1, x_2 \in Q$ and $s > r > n > 0$. Now using the hypothesis $\Gamma_{A\times G} \cong \Gamma_H$, we have

$$p^2a - pa = q^rb - q^nb \tag{1}$$

and $p^3a - pa = q^sb - q^nb$. Thus $q^sb - q^nb = (p^2a - pa)(p + 1) = (q^rb - q^nb)(p + 1)$ and so $(p + 1)(q^r - q^n) = q^s - q^n$. It follows that

$$p\bigl(q^r - q^n\bigr) = q^s - q^r \tag{2}$$

and so $p(q^{r-n} - 1) = q^{s-n} - q^{r-n} = q^{r-n}(q^{s-r} - 1)$. Thus q^{r-n} divides p and so $p = q$ and $r - n = 1$. Now (2) implies that $p(p^{n+1} - p^n) = p^s - p^{n+1}$, so $p^{n+2} = p^s$. Therefore $s = n + 2$. Now (1) yields that $a = p^{n-1}b$ and since $|H| = bp^{n+2} = (bp^{n-1})p^3 = ap^3$, we have $|A \times G| = |H|$. This completes the proof. □

We need the following result about $\mathrm{PSL}(2, q)$.

Proposition 3.21. *Let $G = \mathrm{PSL}(2, q)$, where q is a p-power (p prime) and let $k = \gcd(q - 1, 2)$. Then*

(1) *a Sylow p-subgroup P of G is an elementary abelian group of order q and the number of Sylow p-subgroups of G is $q + 1$.*
(2) *G contains a cyclic subgroup A of order $t = (q - 1)/k$ such that $N_G(\langle u\rangle)$ is a dihedral group of order $2t$ for every non-trivial element $u \in A$.*
(3) *G contains a cyclic subgroup B of order $s = (q + 1)/k$ such that $N_G(\langle u\rangle)$ is a dihedral group of order $2s$ for every non-trivial element $u \in B$.*
(4) *The set $\{P^x, A^x, B^x \mid x \in G\}$ is a partition of G. Suppose a is a non-trivial element of G.*
(5) *If $q > 5$ and $q \equiv 1 \bmod 4$, then*

$$C_G(a) = \begin{cases} N_G(\langle a\rangle) & \text{if } a^2 = 1 \text{ and } a \in A^x \text{ for some } x \in G,\\ A^x & \text{if } a^2 \neq 1 \text{ and } a \in A^x \text{ for some } x \in G,\\ B^x & \text{if } a \in B^x \text{ for some } x \in G,\\ P^x & \text{if } a \in P^x \text{ for some } x \in G.\end{cases}$$

(6) *If $q > 5$ and $q \equiv 3 \bmod 4$, then*

$$C_G(a) = \begin{cases} N_G(\langle a\rangle) & \text{if } a^2 = 1 \text{ and } a \in B^x \text{ for some } x \in G,\\ B^x & \text{if } a^2 \neq 1 \text{ and } a \in B^x \text{ for some } x \in G,\\ A^x & \text{if } a \in A^x \text{ for some } x \in G,\\ P^x & \text{if } a \in P^x \text{ for some } x \in G.\end{cases}$$

(7) *If $q \equiv 0 \bmod 4$, then*

$$C_G(a) = \begin{cases} A^x & \text{if } a \in A^x \text{ for some } x \in G, \\ B^x & \text{if } a \in B^x \text{ for some } x \in G, \\ P^x & \text{if } a \in P^x \text{ for some } x \in G. \end{cases}$$

Proof. Parts (1)–(4) are exactly [19, Satz 8.2, p. 191; Satz 8.2, 8.3, p. 192; Satz 8.5, p. 193]. For other parts, first note that by part (4), the element a lies in the conjugate of one the subgroups P, A or B, and since $C_G(a)^x = C_G(a^x)$, one may assume that a lies in the subgroups P, A or B. If $a \in P$ then with an easy computation, one can see that $C_G(a) = P$. For other cases, in view of the parts (2) and (3), it is enough to note that in the dihedral group D_{2m} of order $2m$ ($m > 2$), if m is odd, then $Z(D_{2m}) = 1$ and if m is even, then $Z(D_{2m})$ is a cyclic group of order 2. □

Lemma 3.22. *Let $G = \mathrm{PSL}(2, q)$. If $A = \{|C_G(x)|\colon x \in G\backslash\{1\}\}$, then*

$$A = \begin{cases} \{q, (q+1)/2, (q-1)/2, q-1\} & \text{if } q \equiv 1 \bmod 4 \text{ and } q \geqslant 7, \\ \{q, (q+1)/2, (q-1)/2, q+1\} & \text{if } q \equiv 3 \bmod 4 \text{ and } q \geqslant 7, \\ \{q, q+1, q-1\} & \text{if } q \equiv 0 \bmod 4, \\ \{3, 4, 5\} & \text{if } q = 5, \\ \{3, 4\} & \text{if } q = 3, \\ \{2, 3\} & \text{if } q = 2. \end{cases}$$

Proof. Since $\mathrm{PSL}(2,2) \cong S_3$, $\mathrm{PSL}(2,3) \cong A_4$ and $\mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5) \cong A_5$ (see [19, Satz 6.14, p. 183]), it is easy to compute the set A for these groups and for $q \geqslant 7$ it follows from Proposition 3.21. □

Proposition 3.23. *Let $G = \mathrm{PSL}(2, q)$. If $\Gamma_G \cong \Gamma_H$ for some group H, then $|G| = |H|$.*

Proof. Note that $Z(G) = 1$, so it is enough to show that $Z(H) = 1$. By Lemma 3.22, there exist two non-trivial elements $x, y \in G$ such that $|C_G(x)| - |C_G(y)| = 1$. By Lemma 3.1, H is a finite group and $|Z(H)|$ divides 1. This completes the proof. □

It is natural to ask if $\Gamma_G \cong \Gamma_H$ and G is a non-abelian finite nilpotent group, is it true that H is nilpotent? We prove this under an additional condition.

Theorem 3.24. *Let G be a finite non-abelian nilpotent group and H be a group such that $\Gamma_G \cong \Gamma_H$ and $|G| = |H|$. Then H is nilpotent.*

Proof. By Lemma 3.1, H is also a finite non-abelian group. If K is any finite group and i a positive integer, then we denote by $m_i(K)$ the number of conjugacy classes with size i of K. By the main result of [13], it is enough to prove that $m_i(G) = m_i(H)$ for any positive integer i. Since $\Gamma_G \cong \Gamma_H$ and $|G| = |H|$, $|Z(G)| = |Z(H)|$ and so $m_1(G) = m_1(H)$. Thus suppose that $i > 1$, $m_i = m_i(G)$ and $X_1, \dots, X_{m_i}$ be all of the conjugacy classes of G with size i. Also assume that $m_i(H) = m_i'$ and $Y_1, \dots, Y_{m_i'}$ be all of the conjugacy classes of H with size i. Note that $X = \{x \in G \mid |G : C_G(x)| = i\} = \{x \in G \mid d(x) = |G| - \frac{|G|}{i}\}$ and $Y = \{y \in H \mid |H : C_H(y)| = i\} =$

$\{y \in H \mid d(y) = |H| - \frac{|H|}{i}\}$. Also since $|G| = |H|$ and $\Gamma_G \cong \Gamma_H$, we have $|X| = |Y|$. But $|X| = m_i i$ and $|Y| = m_i' i$, which imply that $m_i = m_i'$. This completes the proof. □

Now we pose the following conjecture.

Conjecture 3.25. *Let G be a finite non-abelian nilpotent group and H be a group such that $\Gamma_G \cong \Gamma_H$. Then H is nilpotent.*

We need the following result on $\mathrm{GL}(2, q)$ in the sequel.

Proposition 3.26. *Let $G = \mathrm{GL}(2, q)$, where $q = p^n > 2$ (p is a prime) and a be a non-central element of G. Then $C_G(a)$ is conjugate to one of the following subgroups:*

(1) *D the subgroup of all diagonal matrices in G. Also $|D| = (q-1)^2$.*
(2) *I a cyclic subgroup of G, where $|I| = q^2 - 1$.*
(3) *$PZ(G)$, where P is the Sylow p-subgroup of G containing all matrices as $\left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right]$, where x is in the ground field. Also $|PZ(G)| = q(q-1)$.*

Each of the above subgroups is equal to the centralizer of an element in G and G is the union of all conjugates of the above subgroups.

Proof. By [19, Satz 7.2 of Chapter II], we have $|D| = (q-1)^2$ and the number of conjugates of D is $q(q+1)/2$. It is easy to see that $C_G(d) = D$ for any non-central element d of D. By [19, Satz 7.3 of Chapter II], I is a cyclic subgroup of G of order $q^2 - 1$ and $C_G(I) = I$, thus I is the centralizer of any of its generators. Also the number of conjugates of I is $q(q-1)/2$. Thus each conjugate of I or D is also a centralizer of an element of G. By Lemma 3.5, G is an AC-group, so Lemma 3.6 implies that $I^g \cap I^h = D^a \cap D^b = I^g \cap D^a = Z(G)$ for arbitrary elements $a, b, g, h \in G$ such that $I^h \neq I^g$ and $D^a \neq D^b$. Now we prove that the intersection of any two conjugates of $PZ(G)$ is equal to $Z(G)$. Let g and h be two elements such that $gh^{-1} \notin N_G(PZ(G))$ and suppose that $1 \neq x \in P^g$ and $1 \neq y \in P^h$, thus $Q_1 = (PZ(G))^g \leqslant C_G(x)$ and $Q_2 = (PZ(G))^h \leqslant C_G(y)$, since P is an abelian group. Note that $x, y \notin Z(G)$, since $\gcd(|Z(G)|, |P|) = 1$. If $C_G(x) = C_G(y)$, since $C_G(x)$ is abelian, then $C_G(x)$ has a unique Sylow p-subgroup, so $P^g = P^h$, a contradiction. Therefore $C_G(x) \cap C_G(y) = Z(G)$. It now follows that $Q_1 \cap Q_2 = Z(G)$. By a similar argument, one can prove that $Q_1 \cap I^v = Q_1 \cap D^w = Z(G)$ for any two elements $v, w \in G$. Also $D \leqslant N_G(PZ(G)) = N_G(P)$, thus $PZ(G)$ has at most $q+1$ conjugates in G. On the other hand, $\mathrm{PSL}(2, q)$ has $q+1$ Sylow p-subgroups (see [19, Satz 8.2 of Chapter II]). Therefore $PZ(G)$ has exactly $q+1$ conjugates in G. Now we prove that

$$G \backslash Z(G) = \left(\bigcup_{g \in G} (PZ(G))^g \backslash Z(G) \right) \cup \left(\bigcup_{g \in G} I^g \backslash Z(G) \right) \cup \left(\bigcup_{g \in G} D^g \backslash Z(G) \right).$$

The size of the right-hand side of the latter equality is

$$(q+1)(q-1)^2 + (q-1)q^2(q-1)/2 + (q-1)(q-2)q(q+1)/2$$

and also the left-hand side has the size $q^4 - q^3 - q^2 + 1$ as the right-hand side. So the equality holds and it implies that G is the union of all conjugates of the subgroups $PZ(G)$, I and D. It remains to prove that $PZ(G)$ is the centralizer of an element of G. Let x be a non-trivial element of P. Thus $PZ(G) \leqslant C_G(x)$. If there is an element $a \in C_G(x) \backslash PZ(G)$, then $a \in I^g$, $a \in D^h$ or $a \in (PZ(G))^k$ for some $g, h, k \in G$. Now using Remark 3.7 it is easy to see that in any case we can get a contradiction. Thus $PZ(G) = C_G(x)$, as required. This completes the proof. □

Lemma 3.27. *Let G be a finite group and k be the number of conjugacy classes of G. Then $2|E(\Gamma_G)| = |G|^2 - k|G|$.*

Proof. The number $2|E(\Gamma_G)|$ is the number of ordered non-commuting pairs of G, and by [18], it is equal to $|G|^2 - k|G|$. □

Now by counting $E(\Gamma_{\mathrm{GL}(2,q)})$, one can compute the number of conjugacy classes of $\mathrm{GL}(2, q)$, which is a well-known result.

Corollary 3.28. *The number of conjugacy classes of* $\mathrm{GL}(2, q)$ *is* $q^2 - 1$.

Proof. Let $G = \mathrm{GL}(2, q)$. First we count $E(\Gamma_G)$ using the partition given for G in Proposition 3.26. Let P, I and D be the subgroups given in Proposition 3.26. Let $T_1 = PZ(G)$, $T_2 = I$ and $T_3 = D$. Then $\{T_1^g \backslash Z(G), T_2^g \backslash Z(G), T_3^g \backslash Z(G)\colon\ g \in G\}$ is a partition for $G\backslash Z(G)$. We have $|\{T_i^g \backslash Z(G)\colon\ g \in G\}| = |G : N_G(T_i)| = s_i$ for $i = 1, 2, 3$. If $t_i := |T_i^g \backslash Z(G)|$, then

$$2\left|E(\Gamma_G)\right| = 2\left(\sum_{i=1}^{3} \binom{s_i}{2} t_i^2 + \sum_{\substack{i<j\\ i,j=1}}^{3} s_i t_i s_j t_j\right).$$

By the proof of Proposition 3.26, we have $t_1 = (q-1)^2$, $t_2 = (q-1)q$, $t_3 = (q-1)(q-2)$, $s_1 = q+1$, $s_2 = q(q-1)/2$, $s_3 = q(q+1)/2$. Now Lemma 3.27 implies that the number of conjugacy classes of G is $q^2 - 1$. □

4. Clique number and chromatic number of some groups

Lemma 4.1. *Let G be a finite non-abelian group. Then $\chi(\Gamma_G)$ is equal to the minimum number of abelian subgroups of G whose union is G. Also $\omega(\Gamma_G) \leqslant \chi(\Gamma_G) \leqslant |G : Z(G)|$.*

Proof. Let k be the minimum number of abelian subgroups of G whose union is G and suppose G is covered by abelian subgroups $A_1, \ldots, A_k$. Then the vertices of Γ_G in A_i are independent. It follows that $\chi(\Gamma_G) \leqslant k$. Now assume that $\chi = \chi(\Gamma_G)$. Thus there exist χ independent subsets $M_1, \ldots, M_\chi$ of vertices of Γ_G whose union is $G\backslash Z(G)$. It follows that the subgroup generated by $\langle M_j, Z(G)\rangle$ is an abelian subgroup of G, for each j. Clearly G is covered by these χ abelian subgroups, so $\chi \geqslant k$.

It is clear that for any graph Γ, $\chi(\Gamma) \geqslant \omega(\Gamma)$. If $|G : Z(G)| = m$ then $G = \bigcup_{i=1}^m a_i Z(G)$ for some $a_1, \ldots, a_m \in G$. Since $Z_i := \langle a_i, Z(G)\rangle$ is abelian for each i, G is covered by m abelian subgroups $Z_1, \ldots, Z_m$. So the first part, implies that $\chi(\Gamma_G) \leqslant |G : Z(G)|$. This completes the proof. □

Lemma 4.2. *Let G be a non-abelian finite AC-group. Then $\omega(\Gamma_G) = \chi(\Gamma_G)$.*

Proof. Assume $\omega = \omega(\Gamma_G)$ and $\chi = \chi(\Gamma_G)$. Let $x_1, \ldots, x_\omega$ be a maximal clique in Γ_G. Thus each x_i is a non-central element of G and so $C_G(x_i)$ is abelian. It is easy to see that $G = \bigcup_{i=1}^{\omega} C_G(x_i)$. Now Lemma 4.1 implies that $\chi = \omega$. □

Proposition 4.3. *Let $G = \mathrm{GL}(2, q)$, where $q > 2$. Then $\omega(\Gamma_G) = \chi(\Gamma_G) = q^2 + q + 1$. Also if $\Gamma_G \cong \Gamma_H$ for some group H, then $|G| = |H|$.*

Proof. It follows from Proposition 3.26 that there exist elements $x_1, \ldots, x_{q^2+q+1}$ in G such that $G = \bigcup_{i=1}^{q^2+q+1} C_G(x_i)$ such that $C_G(x_i) \cap C_G(x_j) = Z(G)$ for any two distinct indices i, j. Since G is an AC-group, it follows from Lemma 3.6(b) that $x_1, \ldots, x_{q^2+q+1}$ are pairwise non-commutative. So $\omega(\Gamma_G) \geqslant q^2 + q + 1$. On the other hand, since G is covered by $q^2 + q + 1$ abelian subgroups, $\omega(\Gamma_G) \leqslant q^2 + q + 1$. So $\omega(\Gamma_G) = \chi(\Gamma_G) = q^2 + q + 1$, by Lemma 4.2.

By Lemma 3.5, G is an AC-group. If $q > 3$, then G is non-solvable and so Proposition 3.14, yields that $|G| = |H|$. Now assume that $q = 3$. Then it follows from Lemma 3.1 and Proposition 3.26 that H is a finite non-abelian group and $|Z(H)|$ divides $|Z(G)| = 2$. By Proposition 3.26, there is $g \in G$ such that $|C_G(g)| - |Z(G)| = |C_H(h)| - |Z(H)|$ and $|C_G(g)| = 8$ for some non-central element $h \in H$. Thus $|C_H(h)| = (3s + 1)|Z(H)|$, where s is an integer such that $s|Z(H)| = 2$, i.e., $s = 1$ or 2. Since $|\frac{C_H(h)}{Z(H)}| = 3s + 1$ divides $|\frac{H}{Z(H)}| = 23s + 1$. It follows that $s = 1$. Hence $|Z(G)| = |Z(H)|$ and so $|G| = |H|$. □

Lemma 4.4.

$$\omega(\Gamma_{\mathrm{PSL}(2,q)}) = \begin{cases} q^2 + q + 1 & \text{if } q > 5, \\ 21 & \text{if } q = 5 \text{ or } 4, \\ 5 & \text{if } q = 3, \\ 4 & \text{if } q = 2. \end{cases}$$

Proof. By [19, Satz 6.14, p. 183], $\mathrm{PSL}(2, 2) \cong S_3$, $\mathrm{PSL}(2, 3) \cong A_4$, so in the case $q = 2, 3$, the computation of $\omega(\Gamma_{\mathrm{PSL}(2,q)})$ is straightforward. Since $\mathrm{PSL}(2, 4) \cong \mathrm{PSL}(2, 5) \cong A_5$ (see [19, Satz 6.14, p. 183]), so we may assume that $q > 5$ or $q = 4$. It follows from parts (4)–(7) of Proposition 3.21, that $\omega(\Gamma_{\mathrm{PSL}(2,q)})$ is equal to the size of the partition set given in part (4) of Proposition 3.21, so by parts (1)–(3) we have

$$\omega(\Gamma_{\mathrm{PSL}(2,q)}) = (q + 1) + \frac{(q + 1)(q - 1)q/k}{2(q - 1)/k} + \frac{(q + 1)(q - 1)q/k}{2(q + 1)/k} = q^2 + q + 1. \qquad \square$$

The following example due to Isaacs [10], shows that the equality $\omega(\Gamma_G) = \chi(\Gamma_G)$ does not hold in general.

Example 4.5. Let S be an extra special group of order 2^{2m+1}, then

(1) $\omega(\Gamma_S) = 2m + 1$.
(2) $|S : Z(S)| = 2^{2m}$.
(3) $\chi(\Gamma_S) \geqslant 2^m + 1$.

5. Groups which can be characterized by their non-commuting graphs

We begin this section with the following easy observation.

Lemma 5.1. *Suppose that G and H are two non-abelian groups. If $\Gamma_G \cong \Gamma_H$, then $\Gamma_{G\times A} \cong \Gamma_{H\times B}$, for any two abelian groups A and B with the same order.*

Proof. Let $\phi : \Gamma_G \to \Gamma_H$ be a graph isomorphism and $\psi : A \to B$ be a bijective map. Then it is easy to see that $\varphi : (g, a) \mapsto (\phi(g), \psi(a))$ is a graph isomorphism between $\Gamma_{G\times A}$ and $\Gamma_{H\times B}$. □

Let p be a prime number and G, H be two non-abelian groups of order p^3 which are not isomorphic. Then it is easy to see that $\Gamma_G \cong \Gamma_H$ which is isomorphic to the complete $(p+1)$-partite graph whose each part has size $p^2 - p$. Therefore, in general it is not true $G_1 \cong G_2$, for two non-abelian groups with $\Gamma_{G_1} \cong \Gamma_{G_2}$. Also if H_1 and H_2 are two non-abelian groups with isomorphic non-commuting graphs, then in general, even it is not true that $H_1' \cong H_2'$, as the groups H_1 and H_2 with the following presentations show,

$$G = \langle x, y, z \mid x^3 = y^3 = z^2 = [x, y] = (zx)^2 = (zy)^2 = 1\rangle,$$
$$H = \langle x, y \mid x^9 = (xy)^2 = y^2 = 1\rangle.$$

Indeed $|G'| = |H'| = 9$, but $G' \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$ and $H' \simeq \mathbb{Z}_9$. Thus it is natural to ask that under which conditions on G or H we have $G \cong H$. Now we want to study the non-commuting graph of the general linear group $\mathrm{GL}(n, q)$ of degree n over the finite field of order q. Note that $\mathrm{GL}(2, 2) \cong S_3$ and in view of Proposition 3.2, its non-commuting graph is unique.

Proposition 5.2. *Let G be a finite non-solvable AC-group such that $\Gamma_G \cong \Gamma_H$. If $|Z(G)|$ is odd, then $G \cong Z(G) \times \mathrm{PSL}(2, 2^n)$, $H \cong Z(H) \times \mathrm{PSL}(2, 2^n)$ and $|Z(G)| = |Z(H)|$.*

Proof. By Lemma 3.9, we have $G \cong A \times \mathrm{PSL}(2, 2^n)$ for some $n > 1$, where $A = Z(G)$, and Proposition 3.14 implies that $|Z(H)| = |A|$ and $|G| = |H| = |A|2^n(2^n - 1)(2^n + 1)$.
Since G is an AC-group and $\Gamma_G \cong \Gamma_H$, H is also an AC-group. By Lemma 4.4, $K := \mathrm{PSL}(2, 2^n)$ contains elements $x_1, \ldots, x_\omega$, where $\omega = 2^{2n} + 2^n + 1$ such that $|C_K(x_i)| = 2^n$ for $1 \leqslant i \leqslant 2^n$ and the centralizers of the other x_i's have orders $2^n - 1$ or $2^n + 1$. Also $K := \bigcup_{i=1}^{\omega} C_K(x_i)$ and $C_K(x_i) \cap C_K(x_j) = 1$ for all $i \neq j$. Let $\phi : \Gamma_G \to \Gamma_H$ be a graph isomorphism. Then $H = \bigcup_{i=1}^{\omega} C_H(y_i)$, where $y_i := \phi((1, x_i))$ and $|C_H(y_1)| = \cdots = |C_H(y_{2^n+1})| = 2^n|A|$ and $C_H(y_i) \cap C_H(y_j) = Z(H)$ for any two distinct indices i and j. If a Sylow 2-subgroup of H is normal, then $C_H(y_1) = C_H(y_2)$, a contradiction. Thus Sylow 2-subgroups of H are not normal in H. If a Sylow 2-subgroup S of H is cyclic, then [35, Proposition 1, p. 435] implies that H contains an abelian normal subgroup N of odd order such that $H = NS$ and no non-trivial element of S commutes with a non-trivial element of N. Thus $|H| = |N||S|$. Since $|S| = 2^n$, $|N| = |A|(2^n - 1)(2^n + 1)$. Since H is an AC-group, Lemma 3.6 implies that $A \leqslant C_H(y_i)$ for some i. But it is a contradiction since $|C_H(y_i)| \in \{2^n|A|,\ (2^n - 1)|A|,\ (2^n + 1)|A|\}$. Therefore every Sylow 2-subgroup of H is neither normal nor cyclic. Since $|Z(H)|$ is odd and H is an AC-group, the centralizer of every element of order 2 is abelian. Now it follows from a result of Suzuki [34] that $H \cong B \times \mathrm{PSL}(2, 2^m)$ for some abelian group B of odd order and positive integer m. Thus $B \cong Z(H)$ and so $|B| = |A|$. Now since $|G| = |H|$, it follows that $m = n$. This completes the proof. □

Corollary 5.3. *If G is a group such that $\Gamma_G \cong \Gamma_{\mathrm{PSL}(2,2^n)}$ for some $n > 0$. Then $G \cong \mathrm{PSL}(2, 2^n)$.*

Proof. If $n = 1$, then $\mathrm{PSL}(2, 2) \cong S_3$. So, in this case, the proof follows from Proposition 3.2. Assume that $n \geqslant 2$, then $\mathrm{PSL}(2, 2^n)$ is a simple AC-group and the proof follows from Proposition 5.2. □

Following Ito [22], we call a non-abelian group G an F-group if for every two non-central elements $x, y \in G$, such that $C_G(x) \neq C_G(y)$, we have $C_G(x) \not\subseteq C_G(y)$ and $C_G(y) \not\subseteq C_G(x)$.

Lemma 5.4. *Let S be a finite non-abelian non-simple F-group. If G is a group such that $\Gamma_G \cong \Gamma_S$, then G is also a non-simple F-group.*

Proof. First G is also an F-group, for if $x, y \in G\backslash Z(G)$ and $C_G(x) \subseteq C_G(y)$, then $\phi(C_G(x)\backslash Z(G)) \subseteq \phi(C_G(y)\backslash Z(G))$, where $\phi : V(\Gamma_G) \to V(\Gamma_S)$ is a graph isomorphism. So $C_S(\phi(x))\backslash Z(S) \subseteq C_S(\phi(y))\backslash Z(S)$. Thus, since S is an F-group, $C_S(\phi(x)) = C_S(\phi(y))$. It follows that $C_G(x)\backslash Z(G) = C_G(y)\backslash Z(G)$ and so $C_G(x) = C_G(y)$. Therefore G is an F-group. Now suppose, for a contradiction, that G is simple. Thus $Z(G) = 1$, and so [22, Proposition 4.2], implies that G is an AC-group. Now Proposition 5.2 and Corollary 5.3 yield that $S \cong G \cong \mathrm{PSL}(2, 2^n)$ for some $n > 1$, which contradicts the non-simplicity of S. Therefore G is not simple. □

Proposition 5.5. *Let $G = \mathrm{Sz}(2^{2m+1})$ be the Suzuki group over the field with 2^{2m+1} elements $m > 0$ (see [20, p. 182]). If H is a group such that $\Gamma_G \cong \Gamma_H$, then $G \cong H$.*

Proof. Let $q = 2^{2m+1}$ and $r = 2^m$. The Suzuki group G contains elements g_1, g_2, g_3, g_4 such that $|C_G(g_1)| = q^2$, $|C_G(g_2)| = q - 1$, $|C_G(g_3)| = q - 2r + 1$ and $|C_G(g_4)| = q + 2r + 1$ (see [20, Theorems 3.10 and 3.11 of Chapter XI]). Therefore $|Z(H)|$ must divide $q + 2r + 1 - (q - 2r + 1) = 4r$ and $q^2 - q + 1$. Since $4r$ divides q, we have $Z(H) = 1$ which implies that $|G| = |H|$. Now by Theorems 3.10 and 3.11 of [20, pp. 192–193], there exist subgroups A, B, F and C such that their conjugates in G form a partition of G, and A, B, C are cyclic and they are centralizers of some elements in G also F is a Sylow 2-subgroup of G and it is also the centralizer of some elements in G. Let $F_i = C_G(f_i)$ $(i = 1, \ldots, t)$; $A_i = C_G(a_i)$ $(i = 1, \ldots, s)$; $B_i = C_G(b_i)$ $(i = 1, \ldots, r)$; and $C_i = C_G(c_i)$ $(i = 1, \ldots, k)$ be the all conjugates of F, A, B and C in G, respectively. Let $\phi : \Gamma_G \to \Gamma_H$ be a graph isomorphism. Now since $|G| = |H|$, we have $|F_i| = |C_H(\phi(f_i))| = q^2$, $|A_i| = |C_H(\phi(a_i))| = q + 2r + 1$, $|B_i| = |C_H(\phi(b_i))| = q - 2r + 1$ and $|C_i| = |C_H(\phi(c_i))| = q - 1$. Also $\mathcal{P} = \{\overline{F}_i = C_H(\phi(f_i)), \overline{A}_i = C_H(\phi(a_i)), \overline{B}_i = C_H(\phi(b_i)), \overline{C}_i = C_H(\phi(c_i))\}$ is a partition of H. Since $|C_H(\phi(f_i))| = q^2$, $C_H(\phi(f_i))$ is a Sylow 2-subgroup of H. Now we prove that H is a (CIT)-group, where a finite group of even order is called a (CIT)-group if the centralizer of any involution is a 2-group. The order of H is even, since $|H| = |G|$. If x is a non-trivial 2-element of H, since $\mathcal{P}$ is a partition of H, x lies in one of the subgroups $\overline{F}_i$, since the orders of other subgroups of this partition are odd. Let $x = \phi(g)$, where $g \in G$, so

$$\phi\big(C_G(g)\backslash\{1\}\big) = C_H(x)\backslash\{1\}. \tag{$*$}$$

Since $x = \phi(g) \in C_H(\phi(f_i))$, $g \in F_i = C_G(f_i)$. Now it follows from the proof of Theorem 3.11 of [20, p. 193], that $C_G(g) \subseteq F_i$. Thus $(*)$ implies that $C_H(x) \subseteq \overline{F}_i$ and so $C_H(x)$ is a 2-group. Thus H is a (CIT)-group. Now we prove that if P and Q are two distinct Sylow 2-subgroups of H, then $P \cap Q = 1$, i.e. Sylow 2-subgroups of H are independent. For this, it is enough to show that $P = \overline{F}_i$ for some i. Let $1 \neq z \in Z(P)$, then $P \subseteq C_H(z)$ and since $C_H(z)$ is a 2-group, we have $P = C_H(z)$. Hence so far we have proved that G is a (CIT)-group and Sylow 2-subgroups of G are independent. Thus Theorem 5 of [35], implies that if S is a Sylow 2-subgroup of H, then we have one of the following: (i) S is normal, (ii) S is cyclic, (iii) S is a generalized quaternion group, (iv) H is a (ZT)-group (see the definition of a (ZT)-group in [35, p. 426]). But S is not abelian, since F_i is not abelian and $\Gamma_{F_i} \cong \Gamma_S$, and since Sylow 2-subgroups of H are independent and the their number is equal to that of G, so S is not normal. Also S is not a generalized quaternion group, since we have $|S| = |F_i| = q^2$ and $\Gamma_{F_i} \cong \Gamma_S$, so $|Z(F_i)| = |Z(S)|$, which is a contradiction, since $|Z(S)| = 2$ and $|Z(F_i)| = q \geqslant 8$. Therefore H is a (ZT)-group. We know that a (ZT)-group is $\mathrm{PSL}(2, 2^n)$ or $\mathrm{Sz}(q')$ where $n > 2$, $q' = 2^{2m'+1}$ and $m' > 0$ (see [33]). Now since H is not an AC-group, Proposition 5.2 implies that $H \cong \mathrm{Sz}(q')$ and since $|H| = |G|$, we have $G \cong H$. This completes the proof. □

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